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Determination of the current distribution for aerial arrays of collinear elements with any number of sidelobes having equal maxima

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RESEARCH DEPARTMENT

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(RA-51)

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Section	Title	Page
	SUMMARY	1
1.	INTRODUCTION	1
2.	THE NORMALIZED RADIATION PATTERN FOR A FOUR-ELEMENT SYSTEM	1
3.	A NORMALIZED RADIATION PATTERN WITH NARROW BEAMWIDTH AND GOOD SIDELOBE SUPPRESSION	2
4.	A GENERAL FORMULA FOR THE 'CHEBYSHEV TYPE' OF NORMALIZED RADIATION PATTERN	3
5.	FURTHER CONSIDERATION OF THE RELATION BETWEEN BEAMWIDTH, SIDELOBE SUPPRESSION RATIO AND NUMBER OF AERIALS	6
6.	SIX SPECIAL CASES	6
7.	EFFECT OF DEPARTING FROM THE CHEBYSHEV TYPE OF NORMALIZED RADIATION PATTERN	7
8.	A SPECIFIC EXAMPLE	7
9.	CONCLUSIONS	8
10.	ACKNOWLEDGEMENTS	8
11.	REFERENCE	8

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SUMMARY

A 'normalized radiation pattern' is derived initially for a system having only four equally-spaced elements. The general nature of a 'normalized radiation pattern' with good sidelobe suppression ratio is then discussed, and the relevance of Chebyshev polynomials is considered. A general formula for the feed currents in each aerial of an equally-spaced collinear array having any number of equal sidelobes is derived. Particular attention is paid to cases where the number of aerials does not exceed 10. Given any two of the basic quantities sidelobe suppression ratio, beamwidth and number of aerials, the third is determined.

The above is in effect a complete specification if the hypothesis is accepted that a 'Chebyshev-type normalized radiation pattern' is the best available. Departures from this type of pattern are briefly considered, and strongly confirm this hypothesis.

1. INTRODUCTION

For any array of collinear elements, a 'normalized radiation pattern' can be determined, from which the horizontal polar diagram for any particular wavelength and scale of spacing of the aerial elements may be immediately deduced. The simplest, and most important, case is that in which all the elements are equally spaced.

This 'normalized radiation pattern' is discussed in Section 2 for a simple system having only four equally-spaced elements. The general nature of a 'normalized radiation pattern' is discussed in Section 3. If the sidelobes are equal and the aerial elements equally spaced, the 'normalized radiation pattern' can be expressed naturally in terms of Chebyshev polynomials. The relevance of these polynomials is discussed, and simple formulae for the sidelobe suppression ratio and the beamwidth are derived.

Explicit formulae of this nature are stated when the number of elements is between 4 and 10, and in Section 4 corresponding formulae are derived explicitly for an arbitrary number of elements. The relation between beamwidth, sidelobe suppression ratio and number of elements is discussed in Section 5. In Section 6, special attention is paid to six specific cases; for two of these, the relative feed currents are explicitly evaluated numerically. Up to this point we have tacitly assumed that a 'Chebyshev-type normalized radiation pattern' was the best available, once suitable values for the parameters at our disposal had been determined. The effect of departure from the 'Chebyshev type' is discussed in Section 7 in terms of the two cases mentioned in Section 6 for which the relative feed currents were explicitly determined. Section 9 gives conclusions; in particular, departure from the 'Chebyshev type of normalized radiation pattern' appears to be generally undesirable.

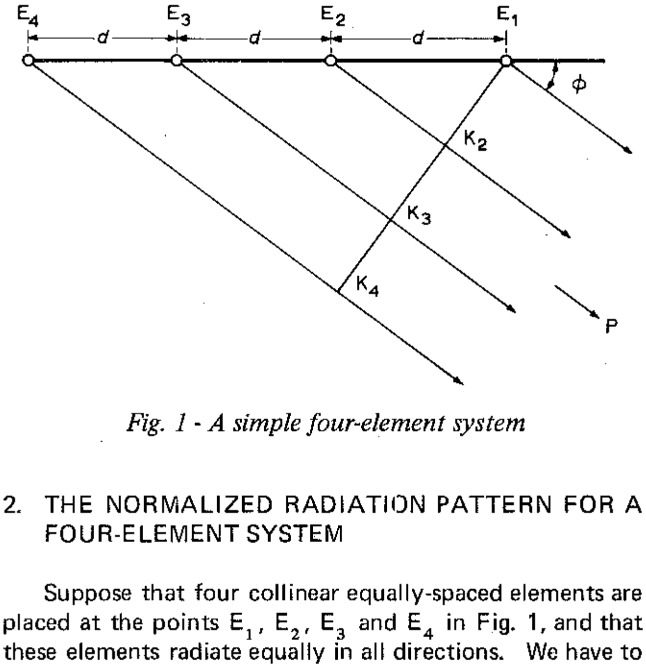

Fig. 1 - A simple four-element system

2. THE NORMALIZED RADIATION PATTERN FOR A FOUR-ELEMENT SYSTEM

Suppose that four collinear equally-spaced elements are placed at the points E_1, E_2, E_3 and E_4 in Fig. 1, and that these elements radiate equally in all directions. We have to consider the way in which the magnitude of the total radiation received at a distant point P varies with its bearing ϕ. If the currents in the elements at E_1, E_2, E_3 and E_4 are progressively phased so that they are respectively I_1, $I_2e^{j\theta}$, $I_3e^{j2\theta}$ and $I_4e^{j3\theta}$ then their separate field contributions received at P will be in the ratio

$$I_1 : I_2e^{j(\theta-\beta d\cos\phi)} : I_3e^{j2(\theta-\beta d\cos\phi)} : I_4e^{j3(\theta-\beta d\cos\phi)}$$

where $\beta = 2\pi/\lambda$, (λ = wavelength). Making the substitution

$$\psi = \theta - \beta d\cos\phi \qquad (1)$$

we may therefore express the total field at P as proportional to $F(\psi)$ where

$$F(\psi) = I_1 + I_2 e^{j\psi} + I_3 e^{j2\psi} + I_4 e^{j3\psi} \qquad (2)$$

Thus we may regard I_1, I_2, I_3 and I_4 as being purely real, both their phases and the effects of the spatial separation of the elements being fully represented by ψ. The radiation pattern as a function of ψ is therefore determined entirely by the amplitude ratios of the currents. In this report this is called the 'normalized radiation pattern' and the parameter ψ is called the 'normalized bearing'. It is shown elsewhere[1] that the progressive phasing (θ) and the element separation (d) may be chosen, independently of these amplitude ratios, so as to use any chosen range of values of ψ (i.e. any segment of the normalized pattern) over the whole range (−180° to +180°) of true bearing, ϕ. Throughout this report, therefore, references to bearing and beamwidth apply to the normalizing parameter ψ.

We are concerned only with the relative magnitude of $F(\psi)$ as ψ varies, and not with its phase; we may therefore disregard factors such as $e^{-j3\psi/2}$ that may appear in the expression for the total field. Putting $Z = e^{j\psi}$ we may write

$$F(\psi) = I_1 Z^{-3/2} + I_2 Z^{-\frac{1}{2}} + I_3 Z^{\frac{1}{2}} + I_4 Z^{3/2} \qquad (3)$$

If there is symmetry so that $I_1 = I_4$ and $I_2 = I_3$, Equation (3) can be further simplified to

$$F(\psi) = 2\left[I_1 \cos(3\psi/2) + I_2 \cos(\psi/2)\right] \qquad (4)$$

Consider first the special case

$$I_1 = I_2 = I_3 = I_4 = I \qquad (5)$$

$F(\psi)$ is zero when $\psi = \frac{1}{2}\pi$, π or $3\pi/2$; it has extreme values when $\psi = 0$ or $\sin^2(\psi/2) = 5/6$. (There is only an inflexion when $\psi = \pi$.) Since $F(\psi + 2\pi) = -F(\psi)$, we need only consider values of ψ between 0 and 2π. When $\cos(\psi/2) = -(1/\sqrt{6})$, $\cos(3\psi/2) = 7/(3\sqrt{6})$ and $|F(\psi)| = 8I/(3\sqrt{6}) = 1{\cdot}0887I$. This quantity is the sidelobe maximum, whereas the main lobe maximum is $4I$, so the sidelobe suppression ratio is $(3\sqrt{6})/2 = 3{\cdot}6742$. It will be relevant later to note that if $\cosh 3\xi_0 = (3\sqrt{6})/2$, then $\cosh \xi_0 = (\sqrt{6})/2$. The sidelobe suppression ratio is somewhat inadequate, and the beamwidth is high. In Section 3 we therefore consider how the coefficients of $|F(\psi)|$ should be chosen so that high sidelobe suppression and low beamwidth can be obtained.

3. A NORMALIZED RADIATION PATTERN WITH NARROW BEAMWIDTH AND GOOD SIDELOBE SUPPRESSION

Let us now suppose that we have a given number N of aerials, and that they are equally spaced, as in Fig. 1, and that the arrangement is symmetrical. Then a repetition of the argument of Section 2 will show that $|F(\psi)|$ can be taken as an expression of the form

$$|F(\psi)| = I_N Z^{(N-1)/2} + I_{N-1} Z^{(N-3)/2} + \ldots + I_2 Z^{-(N-3)/2} + I_1 Z^{-(N-1)/2} \qquad (6)$$

We shall also suppose that all the I's are real.

If N is odd, there will be a middle term independent of Z; if N is even there will not, and because the arrangement is symmetrical $I_r = I_{N+1-r}$ and the right hand of (6) is real for all real ψ. We require $|F(\psi)|$ to be small in the sidelobe region $(\psi \approx \pi)$ but relatively large in the main lobe region $(\psi \approx 0$ or $\psi \approx 2\pi)$. Now in Equation (6) $|F(\psi)|$ is a polynomial of degree $(N-1)$ with real coefficients in the real quantity $(Z^{\frac{1}{2}} + Z^{-\frac{1}{2}})$ or $2\cos\frac{1}{2}\psi$. Now let

$$x = x_0 \cos\tfrac{1}{2}\psi \quad (x_0 \text{ real and} > 1) \qquad (7)$$

so that $|F(\psi)|$ can be expressed also as a polynomial of degree $(N-1)$ with real coefficients in the real variable x. Then for the type of $F(\psi)$ we are seeking, $|F(\psi)|$ will be small when x is sufficiently small $(\psi \approx \pi)$, but relatively large when ψ is small so that x is near its maximum value x_0. This suggests that it may be useful to make $|F(\psi)|$ the Chebyshev polynomial in x of degree $n = N-1$. Clearly multiplication through $F(\psi)$ by a constant is irrelevant, so we shall suppose $|F(\psi)|$ to be normalized in such a way that $|F(\psi)|$ is 1 when $x = 1$. For with this normalization, the Chebyshev polynomial $T_n(x)$ has the remarkable property that $|T_n(x)| \leqslant 1$ in the range $-1 \leqslant x \leqslant 1$. All of the zeros of $T_n(x)$ occur in this range of x, and also all the maxima and minima. All the maxima are equal to +1, and all the minima are equal to −1, and $T_n(1)$ is also equal to 1, while $T_n(-1)$ is equal to $(-1)^n$. More generally, $T_n(-x) = (-1)^n T_n(x)$. This is summed up shortly by saying that if $x = \cos\theta$, $T_n(x) = \cos n\theta$. A very different state of affairs occurs, however, when x exceeds 1. In this case ξ can be determined so that $x = \cosh \xi$, and $T_n(x) = \cosh n\xi$. $T_n(x)$ therefore increases rapidly once x surpasses 1, and this is just what $|F(\psi)|$ is required to do. Chebyshev discovered his polynomials because he sought a polynomial of given degree in x such that given that $T_n(1) = 1$ and that the degree n was fixed, the maximum deviation of $T_n(x)$ from zero in the range $-1 \leqslant x \leqslant +1$ should be a minimum. If therefore we vary the coefficients of $|F(\psi)|$ from those associated with the appropriate Chebyshev polynomial, we can expect a marked deterioration in the sidelobe suppression ratio achieved, with relatively little change in the corresponding beamwidth. This matter is considered further in Section 7.

If

$$|F(\psi)| = \mu T_n(X) \qquad (8)$$

where μ is a real constant, and

$$X = x_0 \cos\tfrac{1}{2}\psi = \tfrac{1}{2}x_0(Z^{\frac{1}{2}} + Z^{-\frac{1}{2}}) \quad (x_0 > 1) \qquad (9)$$

then the coefficients of all powers of Z in Equation (6) are real, so that the contributions of all the elements are in phase; also, the coefficient of any particular power of Z, say Z^k, is the same as that of the inverse power Z^{-k}, so there is symmetry. If $\mu = 1$, the maxima of all the sidelobes are 1, while the maximum of the main lobe is $\cosh n\xi_0$ where $x_0 = \cosh \xi_0$. Hence if R is the suppression ratio

$$R = \cosh n\xi_0 \qquad \text{where } x_0 = \cosh \xi_0 \qquad (10)$$

Furthermore, the first zero of $T_n(x)$ occurs when x =

$\cos(\pi/2n)$, and when ψ is half the beamwidth B. Hence the beamwidth is given immediately by the formula

$$B = 4\cos^{-1}[\cos(\pi/2n)/\cosh \xi_0] \qquad (11)$$

In practice, we shall usually wish to achieve a given beamwidth and sidelobe suppression ratio with as few aerials as possible; that is to say, we shall be given B and R in Equations (10) and (11), and required to determine the corresponding values of x_0 and n. For n above say 8, the term $\cos(\pi/2n)$ in (11) will be very close to unity, and therefore B mainly controls the value of ξ_0 (and hence x_0) required. Equation (10) can then be used to decide what suppression ratio is obtainable for given n or what n is the least with which a given suppression ratio can be obtained. For theoretical understanding of the interactions between B, R, x_0 and n, however, it may be desirable (in some of the cases discussed in Section 6, for example) to regard any two of the four quantities B, R, x_0 and n as given, and to use Equations (10) and (11) to find the remaining two. To find the corresponding feed currents, we need to express $T_n(x_0 \cos\tfrac{1}{2}\psi)$ or $T_n(\tfrac{1}{2}x_0[Z^{\frac{1}{2}} + Z^{-\frac{1}{2}}])$ explicitly in powers of Z. This is done explicitly in the general case in Section 4; Equations (12) to (18) below are for the specific cases n = 3, 4, 5, 6, 7, 8 and 9, and can easily be verified. They are essentially derived by elementary algebraic manipulations from the fact that $T_n(\cos\theta) = \cos n\theta$. We shall use the abbreviation X for $\tfrac{1}{2}x_0(Z^{\frac{1}{2}} + Z^{-\frac{1}{2}})$ and Y_r for $Z^{r/2} + Z^{-r/2}$ (but Y_0 is taken as 1, not 2). If $T_n(X)$ is the relevant polynomial, the number of equally-spaced aerials involved is $(n + 1) = N$. We thus find

$$T_3(X) = \tfrac{1}{2}x_0^3 Y_3 + \frac{3}{2}x_0(x_0^2-1)Y_1 \qquad (12)$$

$$T_4(X) = \tfrac{1}{2}x_0^4 Y_4 + 2x_0^2(x_0^2-1)Y_2 + (x_0^2-1)(3x_0^2-1) \qquad (13)$$

$$\begin{aligned} T_5(X) = \tfrac{1}{2}x_0^5 Y_5 &+ \frac{5}{2}x_0^3(x_0^2-1)Y_3 + \\ &+ \frac{5}{2}x_0(x_0^2-1)(2x_0^2-1)Y_1 \end{aligned} \qquad (14)$$

$$\begin{aligned} T_6(X) = \tfrac{1}{2}x_0^6 Y_6 &+ 3x_0^4(x_0^2-1)Y_4 + \\ &+ \frac{3}{2}x_0^2(x_0^2-1)(5x_0^2-3)Y_2 + \\ &+ (x_0^2-1).(10x_0^4-8x_0^2+1) \end{aligned} \qquad (15)$$

$$\begin{aligned} T_7(X) = \tfrac{1}{2}x_0^7 Y_7 &+ \frac{7}{2}x_0^5(x_0^2-1)Y_5 + \\ &+ \frac{7}{2}x_0^3(x_0^2-1)(3x_0^2-2)Y_3 + \\ &+ \frac{7}{2}x_0(x_0^2-1).(5x_0^4-5x_0^2+1)Y_1 \end{aligned} \qquad (16)$$

$$\begin{aligned} T_8(X) = \tfrac{1}{2}x_0^8 Y_8 &+ 4x_0^6(x_0^2-1)Y_6 + \\ &+ 2x_0^4(x_0^2-1)(7x_0^2-5)Y_4 + \\ &+ 4x_0^2(x_0^2-1).(7x_0^4-8x_0^2+2)Y_2 + \\ &+ (x_0^2-1)(35x_0^6-45x_0^4+15x_0^2-1) \end{aligned} \qquad (17)$$

$$\begin{aligned} T_9(X) = \tfrac{1}{2}x_0^9 Y_9 &+ \frac{9}{2}x_0^7(x_0^2-1)Y_7 + \\ &+ \frac{9}{2}x_0^5(x_0^2-1)(4x_0^2-3)Y_5 + \\ &+ \frac{3}{2}x_0^3(x_0^2-1).(28x_0^4-35x_0^2+10)Y_3 + \\ &+ \frac{9}{2}x_0(x_0^2-1)(14x_0^6-21x_0^4+9x_0^2-1)Y_1 \end{aligned} \qquad (18)$$

In all these formulae, Equations (12) to (18), it will be noticed that the coefficient of the first term written down is $\tfrac{1}{2}x_0^n$, and that every other term written down contains (x_0^2-1) as a factor. This is merely another way of expressing the fact that if

$$x_0 = 1 \quad z = e^{-j\psi} \text{ so that } X = \tfrac{1}{2}x_0(Z^{\frac{1}{2}}+Z^{-\frac{1}{2}}) = \cos\tfrac{1}{2}\psi \qquad (19)$$

then $T_n(X)$ reduces to $\cos(\tfrac{1}{2}n\psi)$. It will also be noticed that all other factors associated with the term in $Y_r = Z^{r/2}+Z^{-r/2}$ are polynomials in x_0^2 which have all their zeros real in x_0^2 and between -1 and $+1$. We are obliged to have $x_0>1$ to achieve satisfactory sidelobe suppression because of Equation (10) and therefore it is not possible to have a 'Chebyshev-type' aerial system in which the aerials are equally spaced and the sidelobes are equal, but terms of the above expansions are missing, so that the corresponding aerials could be omitted.

In Sections 4 and 5 we consider the 'Chebyshev-type' of normalized radiation pattern for a general value of n whereas in Sections 6 and 7 we investigate the hypothesis that any departure from the 'Chebyshev type' of pattern is a serious mistake.

4. A GENERAL FORMULA FOR THE 'CHEBYSHEV TYPE' OF NORMALIZED RADIATION PATTERN

By inspection of Equations (12) to (18), it can be seen that the coefficient of Y_n in $T_n(X)$ is $\tfrac{1}{2}x_0^n$, the coefficient of Y_{n-2} is $\tfrac{1}{2}nx_0^{n-2}(x_0^2-1)$ and the coefficient of Y_{n-4} is $\tfrac{1}{4}nx_0^{n-4}(x_0^2-1)[(n-1)x_0^2-(n-3)]$, but beyond this point generalization is not obvious.

Now Clenshaw* gives the coefficient of X^{n-2k} in $T_n(X)$ in the form

$$(-1)^k\, 2^{n-2k-1}\left\{2\binom{n-k}{k} - \binom{n-k-1}{k}\right\} \qquad (20a)$$

which can also be written in the equivalent form

$$(-1)^k\, 2^{n-2k-1}\frac{n}{k}\binom{n-k-1}{k-1} \qquad (20b)$$

where

$$\binom{N}{K} = \frac{N!}{K!\,(N-K)!} \quad \text{(binomial coefficient)} \qquad (20c)$$

---

* Chebyshev Series for Mathematical Functions, by C.W. Clenshaw, National Physical Laboratory Mathematical Tables, Vol. 5, H.M.S.O., 1962. (See also Research Department Report No. 1969/10.)

and when $\frac{1}{2}x_o(Z^{\frac{1}{2}}+Z^{-\frac{1}{2}})$ is substituted for X, and powers of $(Z^{\frac{1}{2}}+Z^{-\frac{1}{2}})$ are expanded as necessary, $\mathrm{T}_n(X)$ can be reduced to the form

$$\begin{aligned}\mathrm{T}_n(X) = \mathrm{T}_n\left[\tfrac{1}{2}x_o\left(Z^{\frac{1}{2}}+Z^{-\frac{1}{2}}\right)\right] &= \tfrac{1}{2}x_o{}^n Y_n+\frac{n}{2}x_o{}^{n-2}(x_o{}^2-1)Y_{n-2}+\\ &+\frac{nx_o{}^{n-4}}{2.2!}(x_o{}^2-1).\left[(n-1)x_o{}^2-(n-3)\right]Y_{n-4}+\\ &+\frac{nx_o{}^{n-6}}{2.3!}(x_o{}^2-1)\left[(n-1)(n-2)x_o{}^4-2(n-2)(n-4)x_o{}^2+(n-4)(n-5)\right]Y_{n-6}+\\ &+\frac{nx_o{}^{n-8}}{2.4!}(x_o{}^2-1)Y_{n-8}Z+\ldots\ldots\end{aligned} \qquad (21a)$$

where

$$\begin{aligned}Z &= (n-1)(n-2)(n-3)x_o{}^6-3(n-2)(n-3)(n-5)x_o{}^4+\\ &+3(n-3)(n-5)(n-6)x_o{}^2-(n-5)(n-6)(n-7)\end{aligned} \qquad (21b)$$

In (21a), the coefficient of Y_{n-2r} begins with the factor $\frac{1}{2}nx_o{}^{n-2r}(x_o{}^2-1)/r!$ which is multiplied by a polynomial of degree $(r-1)$ in $x_o{}^2$. The coefficient C_k of $x_o{}^{2(r-k-1)}$ in this polynomial can be written

$$C_k = (-1)^k\binom{r-1}{k}\ (n-k-1)(n-k-2)\ \ldots \text{ to } (r-1) \text{ factors}$$

$$[k = 0, 1 \ldots. r-2] \qquad (22a)$$

$$C_{r-1} = (-1)^{r-1}(n-r-1)(n-r-2)\ \ldots\ (n-2r-2)(n-2r-1) \qquad (22b)$$

where in Equation (22a) the factors $(n-k-1)$, $(n-k-2)$ etc. are consecutive and in descending order, except in so far as the factor $(n-r)$ is always omitted. Thus Equation (21) is a crude form of the general formula required, but would be clumsy to handle numerically for large values of n and r, when x_o is not normally a convenient, round number. But suppose we think in terms of expressing the polynomial factor associated with the coefficient of Y_{n-2r} in powers of $(x_o{}^2-1)$ instead of powers of $x_o{}^2$: this is obviously possible. For the coefficient of Y_{n-8}, for example, we want to find A, B, C and D so that

$$\begin{aligned}&(n-1)(n-2)(n-3)x_o{}^6-3(n-2)(n-3)(n-5)x_o{}^4+\\ &+3(n-3)(n-5)(n-6)x_o{}^2-(n-5)(n-6)(n-7)\equiv\\ &\equiv A(x_o{}^2-1)^3+B(x_o{}^2-1)^2+C(x_o{}^2-1)+D\end{aligned} \qquad (23)$$

for all values of x_o, where A, B, C and D may depend on n but not on x_o. By equating coefficients of $x_o{}^6$, it is clear that

$$A = (n-1)(n-2)(n-3) \qquad (24)$$

so if the A-term is subtracted from both sides of (23), we find

$$\begin{aligned}&12[(n-2)(n-3)x_o{}^4-(n-3)(2n-7)x_o{}^2+(n^2-8n+17)]\\ &\qquad\equiv B(x_o{}^2-1)^2+C(x_o{}^2-1)+D\end{aligned} \qquad (25)$$

and therefore

$$B = 12(n-2)(n-3) \qquad (26)$$

Subtracting the B-term from both sides of (25), we have

$$12[3(n-3)x_o{}^2-(3n-11)] \equiv C(x_o{}^2-1) \qquad (27)$$

so that $C = 36(n-3)$ and $D = 24$.

Now this determination of A, B, C and D suggests that a similar process can be carried out with great advantage whatever the value of r may be. 'Wisdom after the event' suggests that if in (23) we had written

$$\left.\begin{aligned}A &= (n-1)(n-2)(n-3)\alpha\\ B &= (n-2)(n-3)\beta\\ C &= (n-3)\gamma\\ D &= \delta\end{aligned}\right\} \qquad (28)$$

then we would have found

$$\alpha = 1; \quad \beta = 12; \quad \gamma = 36; \quad \delta = 24 \qquad (29)$$

so that α, β, γ and δ are independent of n as well as x. Consider therefore whether we can find α_r, β_r, γ_r, δ_r ... to satisfy the identity

$$
\begin{aligned}
&(n-1)(n-2)(n-3)\ \dots\ (n-r+1)x_o^{2r-2}-\\
&-\binom{r-1}{1}(n-2)(n-3)\ \dots\ (n-r+1).(n-r-1)x_o^{2r-4}+\\
&+\binom{r-1}{2}(n-3)(n-4)\ \dots\ (n-r+1)(n-r-1)(n-r-2)x_o^{2r-6}+\\
&\quad+\ \dots.\ +(-1)^{r-1}(n-r-1)(n-r-2)\ \dots.\ (n-2r+1)\\
&\quad\equiv\alpha_r(n-1)(n-2)\ \dots\ (n-r+1)(x_o^2-1)^{r-1}\\
&\qquad+\beta_r(n-2)(n-3)\ \dots\ (n-r+1)(x_o^2-1)^{r-2}\\
&\qquad+\gamma_r(n-3)(n-4)\ \dots\ (n-r+1)(x_o^2-1)^{r-3}\\
&\qquad+\ \dots\dots
\end{aligned}
\tag{30}
$$

where α_r, β_r, γ_r etc. are independent of n and x_o, so that the identity we are seeking, if true at all, is true for *all* values of n and x_o.

Equating coefficients of x_o^{2r-2} gives immediately $\alpha_r = 1$.

Putting $n = 1$ and equating coefficients of x_o^{2r-4} gives

$$\beta_r = r(r-1) = \binom{r}{1}\binom{r-1}{1}\ 1! \tag{31}$$

Putting $n = 2$ and equating coefficients of x_o^{2r-6} gives

$$\gamma_r = \binom{r}{2}\binom{r-1}{2}\ 2! \tag{32}$$

and so on.

We are thus led to suspect that the coefficient η_r of Y_{n-2r} or $Z^{½(n-2r)}+Z^{-½(n-2r)}$ in $T_n[½x_o(Z^{½}+Z^{-½})]$ is given by

$$
\begin{aligned}
\eta_r = \frac{nx_o^{n-2r}(x_o^2-1)}{2r}\Bigg[&\binom{n-1}{r-1}(x_o^2-1)^{r-1}+\binom{r}{1}\binom{n-2}{r-2}(x_o^2-1)^{r-2}+\\
&+\binom{r}{2}\binom{n-3}{r-3}(x_o^2-1)^3+\ \dots.\ +\binom{r}{k}\binom{n-k-1}{r-k-1}(x_o^2-1)^{r-k-1}+\ \dots.\\
&+\binom{r}{3}\binom{n-r+2}{2}(x_o^2-1)^2+\binom{r}{2}\binom{n-r+1}{1}(x_o^2-1)+r\Bigg]
\end{aligned}
\tag{33}
$$

and this is in complete agreement with Equations (12) to (18) for small and moderate values of n. In fact, we can express $T_n[½x_o(Z^{½}+Z^{½})]$ completely by the single equation

$$T_n[½x_o(Z^{½}+Z^{-½})] = ½x_o^n(Z^{\frac{n}{2}}+Z^{\frac{-n}{2}})+½n(x_o^2-1)\xi \tag{34}$$

where

$$
\left.
\begin{aligned}
&\xi = \sum_{r=1}^{\nu}(Z^{\frac{n-2r}{2}}+Z^{\frac{-n-2r}{2}})\frac{x_o^{n-2r}}{r}\sum_{s=o}^{r-1}\binom{r}{\mu}\binom{n-r+s}{s}(x_o^2-1)^s\\
&\nu \text{ is the greatest integer not exceeding } ½n\\
&\mu = 1+s \text{ if } s\leqslant ½r-1\\
&\mu = r-s-1 \text{ if } s\geqslant ½r-1
\end{aligned}
\right\}
\tag{35}
$$

but although (34) is apparently neater and more general than (33), it is much easier to avoid making silly arithmetical errors by using (33) in specific numerical cases.

Thus given any two of the sidelobe suppression ratio R, the beamwidth B, the number of elements $N = n+1$ and the parameter x_o introduced in Equation (7), we can find the other two by means of Equations (10) and (11), and the feed currents in the various elements are proportional to the coefficients η_r defined in Equation (33). (33) is an exact equation, without any restriction on the values of n, r and x_o except that n and r must be integers ($n \geqslant 2r$) and x_o must exceed 1. It is not obvious which of the η_r is the greatest, but Equation (33) is in a much more convenient form than Equation (21), since all the terms in (33) have the same sign and not all of them are necessarily significant, whereas Equation (21) involves differences between nearly equal quantities. In Section 5 we consider the way in which variations in beamwidth, number of aerials and sidelobe suppression ratio interact, and in Section 6, six specific cases. For two of these cases, the actual feed currents are calculated (from (33)) as well as the associated values of beamwidth, number of aerials and sidelobe suppression ratio, and these two cases are considered in Section 7 in relation to the effect of departure from the 'Chebyshev type of normalized radiation pattern'.

5. FURTHER CONSIDERATION OF THE RELATION BETWEEN BEAMWIDTH, SIDELOBE SUPPRESSION RATIO AND NUMBER OF AERIALS

Equations (10) and (11) are two relations involving the basic quantities beamwidth (B), number of elements ($N = n+1$) and sidelobe suppression ratio ($R : 1$). These equations also involve the parameter ξ_o (where $x_o = \cosh \xi_o$) which was introduced to simplify the algebra. With certain approximations, ξ_o can be eliminated to give a single relation between B, R and N; over part of the relevant range of B and R, $\log R$ is linearly related to B for given N. Now if θ is an angle measured in radians and less than 0·7 radians, $\cos\theta \approx 1-\tfrac{1}{2}\theta^2$, and the difference between the two sides of this approximate equation is at worst about 0·01. Likewise if ξ_o is less than 0·5, $1/\cosh \xi_o \approx 1-\tfrac{1}{2}\xi_o^2$ and the greatest discrepancy between the two sides of this approximate equation is also about 0·01. Applying these results with $\theta = B/4$ and ξ_o having the same meaning as in Equation (10), we find that Equation (11) can be reduced to the approximate form

$$B_{(\text{degrees})} \approx \frac{720}{\pi}\left(\xi_o{}^2+4\left[\sin^2\frac{\pi}{4(N-1)}\right]\left\{1-\tfrac{1}{2}\xi_o{}^2\right\}\right)^{1/2} \tag{36}$$

while Equation (10) can be expressed in the alternative form

$$\xi_o = \left[\log_e\left\{R+(R^2-1)^{1/2}\right\}\right]/(N-1) \tag{37}$$

The limits of validity for (36) are that B must be less than about 160° (2·8 radians) and ξ_o less than about 0·5. Furthermore, if R exceeds about 11·6 and $N \geqslant 5$, the first term of the right hand side of (36) is much more important than the second, and $(N-1)\xi_o \approx \log_e(2R)$, so that (37) reduces to

$$\log_{10} R \approx 1{\cdot}895 \times 10^{-3}(N-1)B_{(\text{degrees})} - 0{\cdot}30103 \tag{38}$$

From Equations (36) and (37), or from Equation (38) when it applies, the behaviour of any two of the quantities R, B and N when the third quantity is kept constant can be deduced. R cannot usefully be less than unity.

If R is kept constant, the simplest procedure is to choose a value of N (which must of course be an integer) and deduce ξ_o from (37) and then substitute into (36) to obtain the corresponding value of B. If N is kept constant, a value of R should be chosen and the value of ξ_o deduced from (37); the corresponding value of B is again obtained by substituting into (36). If B is kept constant and a value for N is chosen, (36) becomes a simple equation for $\xi_o{}^2$ after squaring both sides, and R can be deduced from (37), or from (10) in its original form.

In Section 6 we consider six special cases. In particular, cases 1 and 3 indicate how much is gained by increasing the number of aerials by 2 (symmetrically). These cases are also used in Section 7 to indicate the effect of having a normalized radiation pattern which is not of the Chebyshev type.

6. SIX SPECIAL CASES

The following six special cases serve to illustrate further the way in which variations in the three fundamental quantities of beamwidth, sidelobe suppression ratio and number of aerials are related. For cases 1 and 3 the relative values of the feed currents are included in addition to the data for all six cases included in Table 1.

Here cases 1, 4 and 5 have the same suppression ratio, cases 2, 3 and 5 have the same number of aerials, and cases 1 and 3 have the same beamwidth, as do cases 5 and 6. Cases 1 and 2 have a common value of ξ_o, while case 4 is the only one for which there are nine aerials; it is intermediate between cases 1 and 5.

TABLE 1

Particulars of Six Special Cases

Case No.	Number of Aerials (N)	Order of Chebyshev Polynomial $n = N-1$	ξ_o	$x_o = \cosh\xi_o$	Beamwidth (degrees)	Sidelobe Suppression Ratio ($\cosh n\xi_o : 1$)
1	8	7	0·6260	1·2024	143·3	40 : 1
2	10	9	0·6260	1·2024	140·0	144·9 : 1
3	10	9	0·6440	1·2146	143·3	164·5 : 1
4	9	8	0·5477	1·1538	127·1	40 : 1
5	10	9	0·4869	1·1209	114·1	40 : 1
6	8	7	0·4641	1·1096	114·1	12·9 : 1

Cases 1 and 3 indicate the increase obtained in sidelobe suppression (for constant beamwidth) by having two extra aerials. This increase, however, is rapidly dissipated if we try to alter the relative feed currents and thus use a 'non-Chebyshev' type of normalized radiation pattern, as indicated in Section 7 below. The complete expressions for the relative feed currents in cases 1 and 3 respectively are indicated in a manner analogous to Equations (12) to (18) by the formulae:

$$\text{Case 1:}\quad T_7(x) = T_7[0{\cdot}6012(Z^{1/2}+Z^{-1/2})]$$
$$= 1{\cdot}8170(Z^{7/2}+Z^{-7/2})+3{\cdot}9219(Z^{5/2}+Z^{-5/2})+$$
$$+6{\cdot}3406(Z^{3/2}+Z^{-3/2})+7{\cdot}9228(Z^{1/2}+Z^{-1/2}) \qquad (36)$$

$$\text{Case 3:}\quad T_9(x') = T_9[0{\cdot}6073(Z^{1/2}+Z^{-1/2})]$$
$$= 2{\cdot}8765(Z^{9/2}+Z^{-9/2})+8{\cdot}3400(Z^{7/2}+Z^{-7/2})+$$
$$+16{\cdot}4002(Z^{5/2}+Z^{-5/2})+24{\cdot}6588(Z^{3/2}+Z^{-3/2})+$$
$$+29{\cdot}9325(Z^{1/2}+Z^{-1/2}) \qquad (37)$$

If the feed currents, that is, the coefficients of the various powers of Z in Equations (36) and (37), have the actual values given, then the sidelobe maxima are all equal to unity, but the suppression ratio and beamwidth are unaltered if the right hand sides of (36) and (37) are multiplied by any constant.

7. EFFECT OF DEPARTING FROM THE CHEBYSHEV TYPE OF NORMALIZED RADIATION PATTERN

The hypothesis under investigation in this Section is that any departure from the Chebyshev type of normalized radiation pattern discussed above is a mistake. A few simple cases of departure from the Chebyshev type of pattern will be discussed in a straightforward qualitative manner rather than with mathematical rigour. These simple cases indicate, rather than prove, that the above-mentioned hypothesis is strongly supported.

Consider next a normalized radiation pattern of the form

$$f(Z) = T_9[0{\cdot}6073(Z^{1/2}+Z^{-1/2})]+\lambda T_7[0{\cdot}6012(Z^{1/2}+Z^{-1/2})] \qquad (38)$$

Then since the two terms on the right-hand side of (38) are associated with the same beamwidth and therefore the same sidelobe region, we can say immediately that $|f(Z)|$ cannot exceed $1+|\lambda|$ in that sidelobe region. The maximum value of $|f(Z)|$, will not quite reach $1+|\lambda|$, but the shortfall will be small; if λ is positive, the value of Z or $\psi(69{\cdot}24^\circ)$ which makes $1{\cdot}2146 \cos\frac{1}{2}\psi$ equal to unity is close to the value of $\psi(67{\cdot}46^\circ)$ which makes $1{\cdot}2024 \cos\frac{1}{2}\psi$ equal to unity and if λ is negative, the value $163{\cdot}56^\circ$ of ψ which makes $1{\cdot}2146 \cos\frac{1}{2}\psi$ equal to $\cos 4\pi/9$ is also close to the value $158{\cdot}69^\circ$ of ψ which makes $1{\cdot}2024 \cos\frac{1}{2}\psi$ equal to $\cos 3\pi/7$. On the other hand, when $\psi = 0$ and $Z = 1$, the value of $f(Z)$ is $164{\cdot}5+40\lambda$. Hence if λ is positive, the sidelobe suppression ratio is approximately

$$\frac{164{\cdot}5+40\lambda}{1+\lambda} = 164{\cdot}5-\frac{124{\cdot}5\lambda}{1+\lambda} \qquad (39)$$

which decreases sharply as λ increases, and if λ is negative and equal to $-\mu$, the corresponding value of the suppression ratio is

$$\frac{164{\cdot}5-40\mu}{1+\mu} = 164{\cdot}5-\frac{204{\cdot}5\mu}{1+\mu} \qquad (40)$$

which decreases even more sharply as μ increases.

The other possible variation from a Chebyshev type of normalized radiation pattern which appears to be worth considering is one of the form

$$g(Z) = T_9[0{\cdot}6073(Z^{1/2}+Z^{-1/2})]+h(Z) \qquad (41)$$

where $h(Z)$ is so chosen that the terms in $Z^{r/2}$ and $Z^{-r/2}$ are missing for some value of r. At first sight there appears to be some possibility of 'making eight aerials do the work of ten' by this approach, but this appearance is illusory. If we try

$$h(Z) = \lambda T_7[0{\cdot}6012(Z^{1/2}+Z^{-1/2})] \qquad (42)$$

then the values of λ needed to eliminate the terms in $Z^{r/2}$ and $Z^{-r/2}$ are given in Table 2.

TABLE 2

Values of λ in Equation (42) for Reducing a Ten-Aerial System to an Eight-Aerial System

r	7	5	3	1
λ	$-4{\cdot}59$	$-4{\cdot}18$	$-3{\cdot}89$	$-3{\cdot}78$

and Equation (40) shows that the sidelobe suppression ratios associated with these values of λ are miserably small.

The other obvious possibility is to take

$$h(Z) = -8{\cdot}3400(Z^{7/2}+Z^{-7/2}) = -16{\cdot}68 \cos^{7/2}\psi \qquad (43)$$

to eliminate the terms (other than the leading term) of (37) having the smallest coefficient. Here again, it is clear that the maximum value of $|g(Z)|$ given by (41) in the sidelobe region is slightly less then $1+16{\cdot}68$, while the main-lobe maximum is $164{\cdot}5-16{\cdot}68$, so the sidelobe suppression ratio is reduced from $164{\cdot}5$ (for the Chebyshev-type 10-aerial pattern having beamwidth $143{\cdot}3^\circ$) to a mere $8{\cdot}36$. These qualitative results strongly support the hypothesis that any departure from the Chebyshev type of normalized radiation pattern is likely to be highly disadvantageous.

8. A SPECIFIC EXAMPLE

Suppose we have 4 elements and that the sidelobe maxima are required to be 20 dB below the mainlobe maximum. The dipole feed currents are then determined as follows.

In the notation of Section 3, we are given

$$N = 4;\; n = 3;\; R = \text{antilog}_{10}\,(20/20) = 10{\cdot}0 \qquad (44)$$

and from Equations (10) and (11) we deduce directly

$$x_o = 1{\cdot}5403 \qquad B = 223{\cdot}1^\circ \qquad (45)$$

From Equation (6) with $N = 4$, we have

$$F(\psi) = I_4 Z^{3/2} + I_3 Z^{1/2} + I_2 Z^{-1/2} + I_1 Z^{-3/2} \qquad (46)$$

and from Equation (8) we have

$$F(\psi) = \mu \mathrm{T}_3(X)$$

$$= \mu\left[\tfrac{1}{2} x_o^{\,3}\, Y_3 + \frac{3}{2} x_o (x_o^{\,2} - 1) Y_1\right] \text{ from Equation (12)}$$

Y_r was defined as $Z^{r/2} + Z^{-r/2}$, so that

$$F(\psi) = \tfrac{1}{2}\mu x_o^{\,3}\left[Z^{3/2} + Z^{-3/2} + 3(1 - x_o^{\,-2})(Z^{1/2} + Z^{-1/2})\right] \qquad (47)$$

and therefore the right hand sides of (46) and (47) are proportional. Thus

$$I_1 = I_4;\; I_2 = I_3 = 3(1 - x_o^{\,-2}) I_1 = 1{\cdot}736 I_1 \qquad (48)$$

The amplitudes of the dipole feed currents are therefore in the ratio 1 : 1·736 : 1·736 : 1.

This procedure is perfectly general; from Equations (10) and (11) we can always determine those of the quantities B, R, x_o and N which are not given. Thereafter, the relative amplitudes of the dipole feed currents are determined by comparing $|F(\psi)|$, given by Equation (6), with $|F(\psi)|$, given by Equation (8). Equations (12) to (18), or Equation (34) permit us to expand the relevant Chebyshev polynomial in powers of Z. The coefficient of any power of Z, say $Z^{r/2}$, is always the same as the coefficient of $Z^{-r/2}$, and this means that $I_{(N+1+r)/2}$ is always equal to $I_{(N+1-r)/2}$.

It should be noted that, for all the sidelobes to occur at a given level, the amplitudes of the dipole feed currents define the normalized radiation pattern (i.e. the radiation pattern as a function of ψ). It is shown elsewhere[1] that any given segment of the normalized pattern may be used in deciding the directivity of a practical array and that the choice of the segment defines both the separation between the dipoles and the relative phases of their feed currents.

9. CONCLUSIONS

For any given number $N = n + 1$ collinear elements equally spaced, there is a relation between the normalized beamwidth and the sidelobe suppression ratio of the normalized radiation pattern when the sidelobes all have equal maxima. This relation is expressed most simply in the form of Equations (10) and (11). Formulae have been derived from which the relative amplitudes of the currents associated with equal-sidelobe systems can be deduced in the manner indicated in Section 8. Finally the relative phases of the currents, and the separation between the elements, may be chosen as mentioned in Section 2 so as to use any segment of the normalized pattern over the whole range of true bearings. Relations between the variations of beamwidth, sidelobe suppression ratio and number of aerial elements have been examined, and any departure from the Chebyshev type of normalized radiation pattern appears to be highly disadvantageous.

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11. REFERENCE

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